

Representation of Symmetries and Observables in Functional Hilbert Space. I.*

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A representation of symmetry transformations motivated by the functional formulation of quantum field theory is rigorously discussed in a functional Hilbert space. The set of generating functionals is equipped with an inner product by means of the Friedrichs-Shapiro-integral and completed to an Hilbert space. Unitarity, continuity, and reducibility are investigated for the symmetry operations in this space. Also non-unitary transformations are considered.

1. Introduction

The present paper is concerned with a kind of group representation in functional Hilbert space, which is motivated by quantum field theory. It has been proved advantageous to transcribe the non-linear field equations into linear equations for the generating functionals of the time-ordered expectation values^{1,2}. For this purpose it is not necessary to add an external source to the interaction-term, but one has only to be aware, that the fields are operator-valued functionals and thus their expectation values are numerical functionals³. Let

$$\Psi(j) = \int \sum_a \Psi_a(x) j_a(x) d^4x$$

be the field operator smeared with a multi-component test function j then one maps each state $|B\rangle$ into a generating functional by the prescription

$$\begin{aligned} \hat{U}(\varrho) |B\rangle &\rightarrow \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle 0 | T[\Psi(j) \cdots \Psi(j)] \hat{U}(\varrho) |B\rangle \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle 0 | T[\hat{U}^{-1}(\varrho) \Psi(j) \hat{U}(\varrho) \cdots \hat{U}^{-1}(\varrho) \Psi(j) \hat{U}(\varrho)] |B\rangle = \mathfrak{Z}_B(U^+(\varrho) j), \end{aligned}$$

if $|0\rangle$ is stable against $\hat{U}(\varrho)$ and $\hat{U}(\varrho)$ commutes with T . Thus the induced symmetry operation in functional space is

$$\mathfrak{Z}_B(j) \rightarrow \mathfrak{Z}_B(U^+(\varrho) j), \quad (1.3)$$

and makes only use of the representation $U(\varrho)$ in the test function space.

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¹ J. SCHWINGER, Proc. Nat. Acad. Sci. 37, 452 [1951].

$$|B\rangle \rightarrow \mathfrak{Z}_B(j) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle 0 | T \Psi(j) \cdots \Psi(j) |B\rangle \quad (1.1)$$

where $|0\rangle$ denotes the vacuum and T means time-ordering. Assuming the time-ordered products to determine the theory completely $\mathfrak{Z}_B(j)$ contains all the physical information of the state $|B\rangle$.

The map (1.1) not only transforms the field equation into a functional equation but also the symmetry relations for Ψ into corresponding transformations in functional space. Let G be a symmetry group and $U(\varrho)$; $\varrho \in G$, a representation of G in the test function space, then

$$\Psi(U(\varrho) j) = \hat{U}(\varrho) \Psi(j) \hat{U}^{-1}(\varrho) \quad (1.2)$$

expresses the invariance of the theory against G , when $\hat{U}(\varrho)$ are (anti-)unitary operators in the space of the physical states $|B\rangle$. The transition into functional space gives

Let us emphasize that at the present stage of quantum field theory neither $\Psi(j)$ nor the state-space are given in a constructive way. In an ur-field theory⁴ it is just the purpose of the theory to calculate the physical states $|B\rangle$. If it is possible to introduce a suitable scalar product in functional space, allowing beside other things unitary represen-

² J. RZEWUSKI, Field Theory, Vol. 2, Iliffe Books, Ltd., London 1969.

³ R. F. STREATER and A. S. WIGHTMAN, PCT, Spin and Statistics and All That, Benjamin, New York, Amsterdam 1964.

⁴ W. HEISENBERG, Einführung in die einheitliche Feldtheorie der Elementarteilchen, S. Hirzel, Stuttgart 1967.



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tations of the fundamental symmetry groups, one can treat the whole theory in the functional space alone without any reference to the field-theoretic Hilbert space. The solutions of the functional equation can then be identified by applying the infinitesimal group-generators on them, giving now the index B of $\mathfrak{L}_B(j)$ the meaning of a complete set of quantum numbers. That is the point of view formulated in ⁵, where it is also shown that the relevant symmetry transformations really commute with T , which is needed for the derivation of (1.3).

In Chapter 2 we undertake to construct a functional scalar-product in terms of the Friedrichs-Shapiro integral. As integration space serves a real Hilbert space of commuting test functions. That corresponds to the field theory of an Hermitian Boson field operator. The assumption, that the field operator be Hermitian is no restriction in generality, since any field theory can be formulated by means of an Hermitian field operator ^{6, 7}. More delicate is the assumption that the test functions be elements of an Hilbert space. If the basic group would be compact, as is, e. g., the rotation group, the test function space could be chosen — by means of weighted invariant scalar products — “small” enough to allow algebraic singularities in the dual space. Thus, the Schwarz space S can be written as the infinite intersection of always shrinking Hilbert spaces equipped with rotational-invariant scalar products. The dual S' admits algebraic singularities of arbitrary order. Our requirement allows only a finite intersection of Hilbert spaces and thus only algebraic singularities up to a fixed order in the dual space. But this would be sufficient for a renormalizable field theory. Much more complications arise if the fundamental group is non-compact as is the Lorentz group.

In our construction of the functional scalar product it is avoided to define a σ -additive measure on the infinite-dimensional test function space. Integration is merely a linear functional on the space of all generating functionals. Nevertheless, this linear functional is written in measure-theoretic notation, thus expressing explicitly which measures on the finite-dimensional sub-spaces of the test function space correspond to this linear functional. If one

wants the transformation (1.3) to be unitary if $U(\varrho)$ is orthogonal, only Gaussian measures need to be taken into account. Having in this way introduced the scalarproduct an orthonormal basis is provided by the Hermite-polynomials. The transition from polynomials to Hermite-polynomials can be achieved by a functional integral-transformation \mathbf{R} , which plays an important role in subsequent investigations. An alternative realisation of the functional Hilbert space, which is identical with the algebraic construction of ⁸, corresponds in some way to the Lebesgue measure. It is unitary equivalent to the former space.

In Chapter 3 the transformation (1.3) is analyzed without any roundabout way through the field-theoretic Hilbert space. Consequently, (1.3) is considered as defining an operator-function $\pi(U)$, which maps an operator U acting in the test function space into the operator

$$[\pi(U) \mathfrak{L}](j) = \mathfrak{L}(U^+ j) \quad (1.4)$$

acting in the functional Hilbert space. Continuity, unitarity, reducibility, behaviour under the \mathbf{R} -transformation, and the $*$ -homomorphism-property are rigorously discussed in terms of the Friedrichs-Shapiro integration theory.

This paper provides also the mathematical foundations for the investigation of the infinitesimal group-generators and the observables in functional Hilbert space, which is carried through in Part II of the present work. There are also drawn certain physical conclusions, and it is shown how a non-Hermitian field theory fits into the developed framework.

2. The Hilbert Space of Square-integrable Functionals

Let us consider the set of all complex-valued functionals $\mathfrak{L}(j)$ over the real Hilbert space $\mathcal{H}_r \ni j$. Roughly speaking, a functional is a function of infinitely many variables, and one might try to define a functional integral by infinitely iterated integration. In any way, such an integral constitutes a linear functional $E[\mathfrak{L}]$ on the set of functionals \mathfrak{L} . Only this aspect of the functional integral is relevant

⁵ H. STUMPF, K. SCHEERER, and H. G. MÄRTL, Symmetry Conditions on Nonlinear Spinor Field Functionals, to be published in Z. Naturforsch.

⁶ H. P. DÜRR and F. WAGNER, Nuovo Cim. **46**, 223 [1966].

⁷ H. STUMPF, Z. Naturforsch. **25 a**, 575 [1970].

⁸ H. STUMPF, Z. Naturforsch. **24 a**, 188 [1969].

for defining a scalar product in the set of functionals, not the question, if it can be written in terms of a σ -additive measure on \mathcal{H}_r , which indeed gives rise to some complications.

The infinitely iterated integral has to be considered as the limit of finite-dimensional integrals. Therefore, we at first deal with those functionals, which correspond to functions of finitely many variables in a properly chosen basis-system^{9, 10} of \mathcal{H}_r .

Definition 2.1: A functional $\mathfrak{Z}(j)$ is called tame, if there is a projection Q_n on a finite-dimensional subspace $\mathcal{H}_r^{(n)} \subset \mathcal{H}_r$ so that $\mathfrak{Z}(Q_n j) = \mathfrak{Z}(j)$ for all $j \in \mathcal{H}_r$. One then says that $\mathfrak{Z}(j)$ is based on $\mathcal{H}_r^{(n)}$.

When $\mathfrak{Z}(j)$ is based on $\mathcal{H}_r^{(n)}$ it is clearly also based on each larger space $\mathcal{H}_r^{(m)} \supset \mathcal{H}_r^{(n)}$. Given a tame functional based on $\mathcal{H}_r^{(n)}$ we choose such an orthonormal basis $\{e_\nu\}_\infty^1$ of \mathcal{H}_r that $\{e_\nu\}_1^n$ is a basis of $\mathcal{H}_r^{(n)}$ and define the function

$$\begin{aligned} \mathfrak{Z}(x_1, \dots, x_n) &:= \mathfrak{Z}\left(\sum_{\nu=1}^n x_\nu e_\nu\right) = \mathfrak{Z}(Q_n j) \\ &= \mathfrak{Z}\left(\sum_{\nu=1}^\infty x_\nu e_\nu\right) = \mathfrak{Z}(j), \end{aligned} \quad (2.1)$$

where $x_\nu = (e_\nu, j)$.

Suppose now there is given for each n a measure μ_n on R^n . Then it is suggested by itself to define for the tame functional $\mathfrak{Z}(j)$ the functional integral through

$$E[\mathfrak{Z}] := \int_{R^n} \mathfrak{Z}(x_1, \dots, x_n) d\mu_n \quad (2.2)$$

provided that the r.h.s. exists. To make sure, that (2.2) is independent of the subspace on which \mathfrak{Z} is considered to be based, the μ_n have to satisfy a compatibility condition⁹, p. III-4. The requirement, that the μ_n be invariant against all rotations in R^n , singles out from the set of all compatible measures the Gaussian measures defined by

$$d\gamma_n := \exp\left(-\sum_{i=1}^n x_i^2/2c\right) \prod_{i=1}^n dx_i / \sqrt{2\pi c} \quad (2.3)$$

(cf. ⁹, Chap. VI).

The Lebesgue-measures

$$d\lambda_n := \prod_{i=1}^n dx_i \quad (2.4)$$

are also rotational-invariant, but they are not compatible. Nevertheless, we shall show later on that they may be used in some sense to define a functional integral, too. For the moment we consider only the Gaussian measures and for simplicity set $c=1$. From now on (2.2) is specialized to be

$$E[\mathfrak{Z}] = \int_{R^n} \mathfrak{Z}(x_1, \dots, x_n) d\gamma_n. \quad (2.2')$$

At this stage the question arises: in how far does (2.2') depend on our special choice of the basis $\{e_\nu\}$? If we consider a second orthonormal basis $\{e'_\nu\}$ and the corresponding sequence of projections $\{Q'_n\}$, where Q'_n projects on the space spanned by $\{e'_\nu\}_1^n$, it may happen that $\mathfrak{Z}(Q'_n j) \neq \mathfrak{Z}(j)$ for all n , inspite of the fact that $\mathfrak{Z}(j)$ is a tame functional. Does now $E[\mathfrak{Z}(Q'_n j)]$ tend to $E[\mathfrak{Z}(j)]$ for $n \rightarrow \infty$? The affirmative answer is given by a basic theorem of Friedrichs and Shapiro, which we communicate in its strongest version⁹, p. V-8.

Proposition 2.1: Let $\{e_\nu\}$ and $\{e'_\nu\}$ be two orthonormal basis-systems of \mathcal{H}_r and $\{Q_n\}$ and $\{Q'_n\}$ the corresponding sequences of projections. Then for each tame functional $\mathfrak{Z}(j)$ holds

$$\lim_{m, n \rightarrow \infty} \int_{R^{n+m}} |\mathfrak{Z}(Q_n j) - \mathfrak{Z}(Q'_m j)|^2 d\gamma_{n+m} = 0, \quad (2.5)$$

where $\mathfrak{Z}(Q_n j)$ and $\mathfrak{Z}(Q'_m j)$ — after having chosen an orthonormal system in $Q_n \mathcal{H}_r \cup Q'_m \mathcal{H}_r$ — are to be replaced by their corresponding functions in the sense of (2.1) depending of at most $n+m$ variables.

To draw some immediate conclusions of (2.5) we specialize the one basis $\{e_\nu\}$ in such way that $\mathfrak{Z}(Q_n j) = \mathfrak{Z}(j)$ for all $n > n_0$. Observing that the constant unit-functional 1 is integrable with respect to $d\gamma_n$ we deduce from the Schwartz-inequality

$$\lim_{m \rightarrow \infty} \left| \int_{R^{m+n_0}} 1 \cdot [\mathfrak{Z}(j) - \mathfrak{Z}(Q'_m j)] d\gamma_{m+n_0} \right| \leq \lim_{m \rightarrow \infty} \left[\int_{R^{m+n_0}} |\mathfrak{Z}(j) - \mathfrak{Z}(Q'_m j)|^2 d\gamma_{m+n_0} \right]^{1/2} = 0$$

and thus, when $E[\mathfrak{Z}]$ exists,

$$\lim_{m \rightarrow \infty} \int_{R^m} \mathfrak{Z}(Q'_m j) d\gamma_m = \lim_{m \rightarrow \infty} E[\mathfrak{Z}(Q'_m j)] = E[\mathfrak{Z}(j)]. \quad (2.6)$$

⁹ K. O. FRIEDRICHS and H. N. SHAPIRO, Seminar on Integration of Functionals, New York University, Inst. of Math. Sciences 1957.

¹⁰ I. SEGAL, Cargèse Lectures, Gordon and Breach, New York, London, Paris 1967.

Relation (2.6) establishes the complete basis-independence of the linear functional $E[\mathfrak{Z}]$, and we are thus allowed to introduce the basis-free shorthand-notation.

$$E[\mathfrak{Z}(j)] = \int_{\mathcal{H}_r} \mathfrak{Z}(j) \delta\gamma(j) = \int_{\mathcal{H}_r} \mathfrak{Z}(j) e^{-(j,j)/2} \delta j / \sqrt{2\pi}, \quad (2.7)$$

which reminds us of the form which $E[\mathfrak{Z}]$ takes on finite-dimensional subspaces.

Let \mathcal{M} be the algebra of all bounded, continuous, tame functionals, i. e., \mathcal{M} consists of those tame functionals, which are associated with bounded, continuous functions. On \mathcal{M} we introduce the bilinear form

$$\langle \mathfrak{Z}, \mathfrak{S} \rangle := E[\mathfrak{Z}^* \cdot \mathfrak{S}] = \int \mathfrak{Z}^*(j) \mathfrak{S}(j) \delta\gamma(j), \quad \mathfrak{Z}, \mathfrak{S} \in \mathcal{M} \quad (2.8)$$

and define

$$\|\mathfrak{Z}\| := \sqrt{\langle \mathfrak{Z}, \mathfrak{Z} \rangle}. \quad (2.9)$$

The bilinear form (2.8) evidently has all properties of an inner product, especially $\|\mathfrak{Z}\| = 0$ implies that $\mathfrak{Z} = 0$ because of the continuity of \mathfrak{Z} .

Definition 2.2: The completion of \mathcal{M} with respect to the norm (2.9) is called the Hilbert space of square-integrable functionals and denoted by $\mathcal{L}^2(\mathcal{H}_r, \gamma)$.

As a consequence of the fact that the integration space \mathcal{H}_r has infinite dimension there are elements in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$ which cannot be represented as (strict) functionals. Nevertheless, we call each element of

$\mathcal{L}^2(\mathcal{H}_r, \gamma)$ a square-integrable functional. Conversely, it may happen, that a sequence of tame functionals converges for each $j \in \mathcal{H}_r$ without converging in the norm¹¹.

Definition 2.3: Let \mathcal{H}'_r be the space of all complex-valued, linear, continuous functionals on \mathcal{H}_r and B a subset of \mathcal{H}'_r . Then \mathcal{P}_B denotes the algebra of all linear combinations with complex coefficients of the monomials $\prod_{\mu=1}^m f_\mu(j)^{k_\mu}$, $f_\mu \in B$.

For $\mathcal{P}_{\mathcal{H}'_r}$ we simply write \mathcal{P} .

\mathcal{H}'_r is isomorphic to $\mathcal{H}_r + i\mathcal{H}_r = \mathcal{H}_c$, the complex extension of \mathcal{H}_r . For each $f_\mu(j) \in \mathcal{H}'_r$ we have a $f_\mu \in \mathcal{H}_c$ so that $f_\mu(j) = (f_\mu, j)$. The monomials $\prod_{\mu=1}^m f_\mu(j)^{k_\mu} = \prod_{\mu=1}^m (f_\mu, j)^{k_\mu}$ are not elements of \mathcal{M} but, nevertheless, are square-integrable. For (f, j) we have, e. g.,

$$\begin{aligned} \|(f, j)\|^2 &= \int (j, f) (f, j) \delta\gamma(j) \\ &= \int (x_1 e_1 + x_2 e_2, f) (f, x_1 e_1 + x_2 e_2) e^{-(x_1^2 + x_2^2)/2} \frac{dx_1}{\sqrt{2\pi}} \frac{dx_2}{\sqrt{2\pi}} \\ &= (f, e_1) (e_1, f) + (f, e_2) (e_2, f) = (f, f), \end{aligned} \quad (2.10)$$

where $\{e_r\}_1^2$ is a basis of the at most two-dimensional subspace on which (f, j) is based.

Without entering into details let us communicate a statement which results directly from a theorem of SEGAL¹², p. 120.

Proposition 2.2: Let $\{e_r\}$ be an orthonormal basis in \mathcal{H}_r , then $\mathcal{P}\{e_r\}$ is dense in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$.

Because of $\mathcal{P} \supset \mathcal{P}\{e_r\}$, \mathcal{P} is also dense in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$. To enumerate the monomials in $\mathcal{P}\{e_r\}$ we consider the set K consisting of all sequences $\{k_r\} = k$, where the k_r are nonnegative integers which fulfill

$$\sum_{r=1}^{\infty} k_r =: |k| < \infty.$$

The set $K = \{k, |k| < \infty\}$ is countable and may be ordered in some way or the other. We define

$$\mathfrak{P}(k, j) := \prod_{r=1}^{\infty} (k_r!)^{-1/2} (e_r, j)^{k_r}, \quad k \in K. \quad (2.11)$$

Since the dense set $\mathcal{P}\{e_r\}$ is spanned by the countable set $\{\mathfrak{P}(k, j), k \in K\}$ the functional Hilbert space $\mathcal{L}^2(\mathcal{H}_r, \gamma)$ is separable.

The sequence $\{\mathfrak{P}(k, j), k \in K\}$ is not orthonormal. In order to construct an orthonormal basis in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$ we study the functional integral-operator

¹¹ L. GROSS, in: Analysis in Function Space, M.I.T. Press, Cambridge, Mass., 1964.

¹² I. SEGAL, Trans. Amer. Math. Soc. **81**, 106 [1956].

$$(\mathbf{R} \mathfrak{T})(j) := \int \mathfrak{T}(j + i h) e^{-(h, h)/2} \delta h / \sqrt{2\pi} \\ = \langle 1, \mathfrak{T}(j + i h) \rangle_h. \quad (2.12)$$

\mathbf{R} is evidently well defined for all $\mathfrak{T}(j) \in \mathcal{P}$, where it is also clear how to read $\mathfrak{T}(j + i h)$, since the $\mathfrak{T}(j)$ depend on j only through the complex scalar product (f_μ, j) . [To get holomorphic polynomials we should take care that the complex test functions appear always in the right place of (f_μ, j) ; cf. also Part II.] We get now for our monomials $\mathfrak{P}(k, j)$

$$\mathbf{R} \mathfrak{P}(k, j) = \int \prod_v (k_v!)^{-1/2} (e_v, j + i h)^{k_v} \delta \gamma(h) \\ = \prod_v (k_v!)^{-1/2} \int (x_v + i y_v)^{k_v} e^{-y_v^2/2} dy_v / \sqrt{2\pi} \\ = \prod_v (k_v!)^{-1/2} H e_{k_v}((e_v, j)) \quad (2.13)$$

where $(e_v, j) = x_v$, $(e_v, h) = y_v$, and the integral-representation of the Hermite polynomials¹³ $H e_{k_v}(x_v)$ leads to the last identity.

Proposition 2.3: Define $\mathfrak{S}(k, j) = \mathbf{R} \mathfrak{P}(k, j)$. Then $\{\mathfrak{S}(k, j), k \in K\}$ is an orthonormal basis in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$.

Proof: Orthonormality follows from (2.13) immediately by the orthonormality of the Hermite polynomials¹³, p. 106, i. e.,

$$\langle \mathfrak{S}(k, j), \mathfrak{S}(k', j) \rangle = \prod_v (k_v! k'_v!)^{-1/2} \int H e_{k_v}(x_v) H e_{k'_v}(x_v) e^{-x_v^2/2} (dx_v / \sqrt{2\pi}) = \prod_v \delta_{k_v, k'_v} =: \delta_{k, k'}.$$

To show the completeness it is sufficient to prove that $\mathbf{R} \mathcal{P}\{e_v\}$, which is by definition spanned by the $\mathfrak{S}(k, j)$, is dense in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$. Since the $\mathfrak{S}(k, j)$ are itself polynomials, we have $\mathbf{R} \mathcal{P}\{e_v\} \subset \mathcal{P}\{e_v\}$. Using a formula of the Gauss-transformation¹³, p. 185,

$$\int_{-\infty}^{+\infty} e^{-(x-y)^2} H e_n(x/\sqrt{2}) dx = \sqrt{\pi} (y/\sqrt{2})^n$$

one obtains

$$\int_{-\infty}^{+\infty} H e_n(x+y) e^{-x^2/2} dx / \sqrt{2\pi} = y^n,$$

so that

$$\mathbf{R}^{-1} \mathfrak{S}(k, h) = \int \mathfrak{S}(k, j + h) e^{-(j, j)/2} \delta j / \sqrt{2\pi} = \mathfrak{P}(k, h). \quad (2.14)$$

The integral-operator \mathbf{R}^{-1} is evidently defined on \mathcal{P} . Let us introduce for every complex α the dilatation

$$\mathbf{D}_\alpha \mathfrak{T}(j) := \mathfrak{T}(\alpha j), \quad (2.15)$$

which leaves the set $\mathcal{P}\{e_v\}$ invariant and which satisfies $\mathbf{D}_{1/\alpha} = \mathbf{D}_\alpha^{-1}$ on $\mathcal{P}\{e_v\}$. Then \mathbf{R}^{-1} can be written as $\mathbf{D}_i \mathbf{R} \mathbf{D}_{-i}$. Therefore, $\mathbf{R}^{-1} \mathcal{P}\{e_v\} \subset \mathcal{P}\{e_v\}$ and thus $\mathcal{P}\{e_v\} \subset \mathbf{R} \mathcal{P}\{e_v\}$, so that $\mathbf{R} \mathcal{P}\{e_v\} = \mathcal{P}\{e_v\}$ and therefore is dense in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$, q.e.d.

For later use we consider the operator \mathbf{N} defined by

$$\mathbf{N} \mathfrak{P}(k, j) := |k| \mathfrak{P}(k, j). \quad (2.16)$$

\mathbf{N} possesses only nonnegative, integer eigenvalues but is not Hermitian. By $\mathcal{L}_n^2(\mathcal{H}_r, \gamma)$ let us denote the eigenspace of \mathbf{N} to the eigenvalue n . The \mathbf{R} -transformed operator $\bar{\mathbf{N}} = \mathbf{R} \mathbf{N} \mathbf{R}^{-1}$ has the property

$$\bar{\mathbf{N}} \mathfrak{S}(k, j) = |k| \mathfrak{S}(k, j) \quad (2.17)$$

and is Hermitian, so that its eigenspaces $\bar{\mathcal{L}}_n^2(\mathcal{H}_r, \gamma)$ are mutually orthogonal and the functional Hilbert space can be decomposed into

$$\mathcal{L}^2(\mathcal{H}_r, \gamma) = \sum_{n=0}^{\infty} \bar{\mathcal{L}}_n^2(\mathcal{H}_r, \gamma).$$

It is sometimes customary to shift the exponential $\exp[-(j, j)/2]$ from the measure $\delta \gamma(j)$ to the functionals $\mathfrak{T}(j)$. From our point of view we should say that in this case the exponential does no longer belong to the definition of the integration-functional $E[\mathfrak{T}(j)]$. This procedure can be considered as a mapping into another functional Hilbert space. For this purpose we define the operator \mathbf{V} by

$$\mathbf{V} \mathfrak{S}(k, j) := e^{-(j, j)/2} \mathbf{D}_{\sqrt{2}} \mathfrak{S}(k, j) =: \mathfrak{S}(k, j). \quad (2.18)$$

¹³ W. MAGNUS und F. OBERHETTINGER, Formeln und Sätze Springer-Verlag, Berlin 1948.

für die speziellen Funktionen der mathematischen Physik,

In the linear span of the $\mathfrak{S}(k, j)$ we introduce the inner product

$$\begin{aligned} \langle \mathfrak{S}(k, j), \mathfrak{S}(k', j) \rangle_\lambda &:= \int \mathfrak{S}^*(k, j) \mathfrak{S}(k', j) \delta\lambda(j) \\ &= \int \mathfrak{S}^*(k, j) \mathfrak{S}(k', j) \delta j / \sqrt{\pi} = \int \mathfrak{S}^*(k, j) \mathfrak{S}(k', j) e^{-(j, j)/2} \delta j / \sqrt{2\pi} = \delta_{k, k'}, \end{aligned} \quad (2.19)$$

where the integrals and the relations between them have only a definite meaning in the finite-dimensional subspaces of \mathcal{H}_r .

Definition 2.4: The completion of the linear span generated by the $\mathfrak{S}(k, j)$, $k \in K$, with respect to the scalar-product (2.19) is denoted by $\mathcal{L}^2(\mathcal{H}_r, \lambda)$.

The connection of $\mathcal{L}^2(\mathcal{H}_r, \lambda)$ with the construction in ⁸ will be made transparent at the end of Chapter 3 of Part II, where the appropriate creation and annihilation operators are investigated. Since \mathbf{V} is by definition a unitary operator $\mathcal{L}^2(\mathcal{H}_r, \lambda)$ is principally as good as $\mathcal{L}^2(\mathcal{H}_r, \gamma)$. We start our investigations of the symmetries and observables, however, always in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$ and then transform the results by means of \mathbf{V} into $\mathcal{L}^2(\mathcal{H}_r, \lambda)$.

3. Group Representations in the Functional Hilbert Space

The preceding Chapter provides us with the mathematical setting necessary to investigate the operator-function

$$[\pi(U) \mathfrak{T}](j) = \pi(U) \mathfrak{T}(j) = \mathfrak{T}(U^+ j), \quad (3.1)$$

where U is an operator in \mathcal{H}_r of a rather general kind. Of course, we have to assume that U is defined on a dense domain $\mathcal{D} \subset \mathcal{H}_r$, which is necessary and sufficient that U^+ exists. Then $\pi(U)$ is defined on all tame functionals $\mathfrak{T}(j)$ based on a subspace $\mathcal{H}^{(m)} \subset \mathcal{D}$ and for which $\mathfrak{T}(U^+ j)$ is square-integrable. For, let

$Q^m = \sum_{\mu=1}^m e_\mu(e_\mu, \cdot)$ project on $\mathcal{H}^{(m)}$ then

$$\|Q_m U^+ j\| = \left\| \sum_{\mu=1}^m e_\mu(e_\mu, U^+ j) \right\| \leq \left(\sum_{\mu=1}^m \|U e_\mu\| \right) \|j\|$$

so that $Q_m U^+$ is bounded and can be extended to the whole of \mathcal{H}_r . Thus $\mathfrak{T}(U^+ j) = \mathfrak{T}(Q_m U^+ j)$ has meaning for all j , how small the domain of U^+ ever may be.

Assuming the range of U_1^+ to be contained in the domain of U_2^+ we have the homomorphism-property

$$\pi(U_1 U_2) \mathfrak{T}(j) = \mathfrak{T}(U_2^+ U_1^+ j) = \pi(U_2) \mathfrak{T}(U_1^+ j) = \pi(U_1) \pi(U_2) \mathfrak{T}(j). \quad (3.2)$$

An important case is that U is orthogonal.

Proposition 3.1: Let U be an orthogonal transformation in \mathcal{H}_r , then $\pi(U)$ is a unitary transformation in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$.

Proof: It is sufficient to show that $\pi(U)$ conserves the norm on the dense domain of the tame functionals. Let $\mathfrak{T}(j)$ be tame and based on $\mathcal{H}^{(n)}$, choose an orthonormal basis $\{e_r\}$ of \mathcal{H}_r in the way that the projection Q_n on $\mathcal{H}^{(n)}$ is represented by $\sum_{r=1}^n e_r(e_r, \cdot)$, then according to (2.1) and (2.2')

$$\|\mathfrak{T}\|^2 = \int_{R^n} |\mathfrak{T}(x_1, \dots, x_n)|^2 d\gamma_n. \quad (i)$$

If we succeed in finding another orthonormal basis $\{e'_r\}$ and a projection $Q'_n = \sum_{r=1}^n e'_r(e'_r, \cdot)$, so that $\pi(U) \mathfrak{T}(j) = \pi(U) \mathfrak{T}(Q'_n j)$ then the norm of the

transformed functional is calculated by

$$\|\pi(U) \mathfrak{T}\|^2 = \int_{R^n} |\pi(U) \mathfrak{T}(y_1, \dots, y_n)|^2 d\gamma_n, \quad (ii)$$

where $\pi(U) \mathfrak{T}((e'_1, j), \dots, (e'_n, j))$ is according to (2.1) equal to $\mathfrak{T}(U^+ \sum_{r=1}^n e'_r(e'_r, j))$. The point is, that $d\gamma_n$ is the same Gaussian measure in (i) and in (ii) by definition of the functional integral. Setting now $Q'_n = U Q_n U^{-1}$ and $e'_r = U e_r$, one indeed gets $\mathfrak{T}(U^+ Q'_n j) = \mathfrak{T}(Q_n U^+ j) = \mathfrak{T}(U^+ j)$ and, moreover,

$$\begin{aligned} (\pi(U) \mathfrak{T})(y_1, \dots, y_n) &= \mathfrak{T}(U^+ \sum_{r=1}^n e'_r(e'_r, j)) \\ &= \mathfrak{T}(\sum_{r=1}^n e_r(e_r, j)) = \mathfrak{T}(y_1, \dots, y_n), \end{aligned}$$

where $y_r = (e'_r, j)$, so that (i) = (ii). There has been

no need for a variable-substitution in an infinite-dimensional integral, q.e.d.

Let us now consider one-parametric, weakly continuous operator-functions U_t . For the discussion of the continuity of $\pi(U_t)$ we make more stringent assumptions on the domain of definition which are, nevertheless, appropriate for our further investigations in Part II. We need for this purpose the definition and some properties of the functional derivative.

The weak functional derivative (Gâteaux-derivative) is defined through ¹⁴, p. 110,

$$\begin{aligned} [(h, \delta/\delta j) \mathfrak{Z}](j) &:= (h, \delta/\delta j) \mathfrak{Z}(j) : \\ &= \lim_{\varepsilon \rightarrow 0} (1/\varepsilon) [\mathfrak{Z}(j + \varepsilon h) - \mathfrak{Z}(j)], \end{aligned}$$

where $h \in \mathcal{H}_r$. We call a tame functional $\mathfrak{Z}(j)$ continuously differentiable if the derivative $(h, \delta/\delta j) \mathfrak{Z}(j)$ — which is tame, too — corresponds to a continuous function and is also continuous in h , i. e.,

$$|(h, \delta/\delta j) \mathfrak{Z}(j)| \leq \mathfrak{S}(j) \sqrt{(h, h)},$$

$\mathfrak{S}(j)$ being a positive functional independent of h . If the last inequality is satisfied, $\mathfrak{Z}(j)$ is also called Fréchet-differentiable ¹⁴, p. 110. For tame, continuously differentiable functionals holds the mean value theorem being nothing but the corresponding theorem of the classical analysis for functions of finitely many variables, which reads in basis-independent notation

$$\mathfrak{Z}(j + h) = \mathfrak{Z}(j) + [(h, \delta/\delta j) \mathfrak{Z}](j + \vartheta h), \vartheta \in [0, 1].$$

In order to ensure that $\mathfrak{Z}(U_t^+ j)$ is in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$ it is not sufficient that $\mathfrak{Z}(j)$ is square-integrable, even

if the U_t^+ are all bounded operators. Consider, for example, the functional $\exp[(j, A j)]$ which for appropriate A may be in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$. $\pi(U_t) \exp[(j, A j)]$, however, is in general not square-integrable. To circumvent such complications let us introduce the following set of well-behaved functionals.

Definition 3.1: Let D be a dense, linear manifold in \mathcal{H}_r . We denote by C_D the set of all tame, continuously differentiable functionals $\mathfrak{Z}(j)$ based on subspaces of D for which

$$|\mathfrak{Z}(B j)| \leq \mathfrak{S}(j)$$

$$\text{and } |(h, \delta/\delta j) \mathfrak{Z}(B j)| \leq \mathfrak{S}'(j) \sqrt{(h, h)},$$

holds for all bounded, linear operators B , in \mathcal{H}_r for which there exists a $b > 1$, so that $\|B\| < b$, and for some positive and square-integrable functionals $\mathfrak{S}(j)$, $\mathfrak{S}'(j)$.

Remark: C_D is dense in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$, since it contains the dense set \mathcal{P}_D .

We are now able to announce

Proposition 3.2: Let U_t be a one-parametric, weakly continuous operator function defined together with the adjoint operators U_t^+ on the common, dense domain $D \subset \mathcal{H}_r$. Then $\pi(U_t)$ is strongly continuous on C_D .

Proof: Without restriction of generality we assume $U_0 = U_0^+ = 1$ and only investigate continuity at $t = 0$. Since U_t is weakly continuous it is also U_t^+ .

(a) Let $\mathfrak{Z}(j) \in C_D$ be based on $\mathcal{H}^{(m)} = Q_m \mathcal{H}_r$, choose an orthonormal basis $\{e_\nu\}$, $e_\nu \in D$, so that

$$Q_m = \sum_{\mu=1}^m e_\mu(e_\mu, \cdot). \text{ Consider}$$

$$\begin{aligned} \lim_{t \rightarrow 0} \langle \mathfrak{Z}, \pi(U_t) \mathfrak{Z} \rangle &= \lim_{t \rightarrow 0} \int \mathfrak{Z}^*(Q_m j) \mathfrak{Z}(Q_m U_t^+ j) \delta\gamma(j) \\ &= \lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} \int \mathfrak{Z}^*(Q_m j) \mathfrak{Z}(Q_m U_t^+ Q_n j) \delta\gamma(j), \end{aligned} \quad (i)$$

where $Q_n = \sum_{\nu=1}^n e_\nu(e_\nu, \cdot) \supset Q_m$ for $n \geq m$. To justify the last identity of (i) one has to show that $\mathfrak{Z}(Q_m U_t^+ j)$ is a tame functional. Denote by $\mathcal{G}^{(m')}$ the space $U_t \mathcal{H}^{(m)}$ and the projection on $\mathcal{G}^{(m')}$ by $Q_{m'}$. For all $j, h \in \mathcal{H}_r$ then holds

$$(h, Q_m U_t^+ j) = (U_t Q_m h, j) = (U_t Q_m h, Q_{m'} j) = (h, Q_m U_t^+ Q_{m'} j).$$

Since $Q_m U_t^+$ is bounded the equality of the matrix-elements is sufficient for the equality of the operators, thus,

$$Q_m U_t^+ = Q_m U_t^+ Q_{m'}$$

proving $\mathfrak{Z}(Q_m U_t^+ j)$ to be tame and based on $\mathcal{G}^{(m')}$.

¹⁴ E. HILLE and R. S. PHILLIPS, Functional Analysis and Semigroups, Amer. Math. Soc., Providence 1957.

For fixed n the integral in (i) is a finite-dimensional Lebesgue-integral majorized for each t by $\int \mathfrak{S}(Q_m j) \mathfrak{S}(Q_m j) \delta\gamma(j) < \infty$. Define $x_r := (e_r, j)$ and $U_{t,\mu}^+ := (e_\mu, U_t^+ e_r)$ then

$$\mathfrak{Z}(Q_m U_t^+ Q_n j) = \mathfrak{Z}\left(\sum_{r=1}^n U_{t,1r}^+ x_r, \dots, \sum_{r=1}^n U_{t,mr}^+ x_r\right)$$

is continuous in t . For fixed n we thus are allowed to perform the t -limit under the integral, i. e.,

$$\lim_{t \rightarrow 0} \int \mathfrak{Z}^*(Q_m j) \mathfrak{Z}(Q_m U_t^+ Q_n j) \delta\gamma(j) = \int \mathfrak{Z}^*(Q_m j) \mathfrak{Z}(Q_m j) \delta\gamma(j),$$

since $Q_m Q_n = Q_m$, $n \geq m$.

(β) We now show, that in (i) the limits may be interchanged proving uniformity of the n -limit with respect to t . It is sufficient to take t from a closed interval $t \in [0, t_0]$, $t_0 > 0$. By the mean value theorem we obtain

$$\begin{aligned} \text{(ii)} &= \left| \int \mathfrak{Z}^*(Q_m j) [\mathfrak{Z}(Q_m U_t^+ j) - \mathfrak{Z}(Q_m U_t^+ Q_n j)] \delta\gamma(j) \right| \\ &\leq \int |\mathfrak{Z}^*(Q_m j) (Q_m U_t^+ P_n j, \delta/\delta j) \mathfrak{Z}(Q_m U_t^+ j + \vartheta Q_m U_t^+ P_n j)| \delta\gamma(j), \end{aligned}$$

where $P_n = 1 - Q_n$. Since $\mathfrak{Z} \in C_D$ we continue the estimation as follows

$$\begin{aligned} \text{(ii)} &\leq \int \mathfrak{S}(Q_m j) \mathfrak{S}'(Q_m j) (Q_m U_t^+ P_n j, Q_m U_t^+ P_n j)^{1/2} \delta\gamma(j) \\ &\leq \left[\int \mathfrak{S}^2(Q_m j) \mathfrak{S}^2(Q_m j) \delta\gamma(j) \right]^{1/2} \left[\int (Q_m U_t^+ P_n j, Q_m U_t^+ P_n j) \delta\gamma(j) \right]^{1/2} \\ &= C \left[\int \sum_{r, r'=n+1}^{\infty} (j, e_r) (e_{r'}, j) (e_r, U_t Q_m U_t^+ e_{r'}) \delta\gamma(j) \right]^{1/2} = C \left[\sum_{r=n+1}^{\infty} (e_r, U_t Q_m U_t^+ e_r) \right]^{1/2} \quad [\text{cf., (2.10)}] \\ &= C \left[\sum_{r=n+1}^{\infty} \sum_{\mu=1}^m (U_t e_\mu, e_r) (e_r, U_t e_\mu) \right]^{1/2} = C \left[\sum_{\mu=1}^m (U_t e_\mu, P_n U_t e_\mu) \right]^{1/2} \leq C \left[\sum_{\mu=1}^m \|U_t e_\mu\|^2 \right]^{1/2} \|P_n\|^{1/2}. \end{aligned}$$

In the last step the boundedness of P_n has been employed. Clearly, $\lim_{n \rightarrow \infty} \|P_n\| = 0$. On the other hand $\left[\sum_{\mu=1}^m \|U_t e_\mu\|^2 \right]^{1/2}$ assumes a maximal value M for $t \in [0, t_0]$. Thus

$$\text{(ii)} \leq C M \|P_n\|^{1/2} \quad \text{and tends to zero uniformly in } t.$$

Interchanging in (i) the limits leads to $\lim_{t \rightarrow 0} \langle \mathfrak{Z}, \pi(U_t) \mathfrak{Z} \rangle = \langle \mathfrak{Z}, \mathfrak{Z} \rangle$. (iii)

(γ) We next investigate

$$\lim_{t \rightarrow 0} \langle \mathfrak{Z}(U_t^+ j), \mathfrak{Z}(U_t^+ j) \rangle = \lim_{t \rightarrow 0} \lim_{n \rightarrow \infty} \langle \mathfrak{Z}(U_t^+ Q_n j), \mathfrak{Z}(U_t^+ Q_n j) \rangle, \quad \text{(iv)}$$

where we have again to justify the interchangeability of the n - and t -limit. Since

$$\begin{aligned} &\langle \mathfrak{Z}(U_t^+ j), \mathfrak{Z}(U_t^+ j) \rangle - \langle \mathfrak{Z}(U_t^+ Q_n j), \mathfrak{Z}(U_t^+ Q_n j) \rangle \\ &= \langle \mathfrak{Z}(U_t^+ j) - \mathfrak{Z}(U_t^+ Q_n j), \mathfrak{Z}(U_t^+ j) \rangle + \langle \mathfrak{Z}(U_t^+ Q_n j), \mathfrak{Z}(U_t^+ j) - \mathfrak{Z}(U_t^+ Q_n j) \rangle \end{aligned}$$

we may reduce this expressions to the case (β) after having replaced the single factors in the two scalar products by $\mathfrak{S}(j)$. Thus, the n -limit in (iv) is uni-form in t and we may invert the order of the limits in (iv) arriving at

$$\lim_{t \rightarrow 0} \langle \pi(U_t) \mathfrak{Z}, \pi(U_t) \mathfrak{Z} \rangle = \langle \mathfrak{Z}, \mathfrak{Z} \rangle. \quad \text{(v)}$$

Combining (iii) with (v) leads to

$$\lim_{t \rightarrow 0} \|\pi(U_t) \mathfrak{Z} - \mathfrak{Z}\|^2 = \lim_{t \rightarrow 0} [\langle \pi(U_t) \mathfrak{Z}, \pi(U_t) \mathfrak{Z} \rangle + \langle \mathfrak{Z}, \mathfrak{Z} \rangle - \langle \pi(U_t) \mathfrak{Z}, \mathfrak{Z} \rangle - \langle \mathfrak{Z}, \pi(U_t) \mathfrak{Z} \rangle] = 0$$

for all $\mathfrak{Z} \in C_D$. q.e.d.

Remark: In the case that U_t is orthogonal the function $\pi(U_t)$ of unitary operators is evidently continuous in the whole of $\mathcal{L}^2(\mathcal{H}_r, \gamma)$. Let \mathfrak{Z} be an arbitrary, square-integrable functional then it may be

strongly approximated by a sequence \mathfrak{Z}_n of functionals from C_D the latter set being dense in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$. Thus,

$$\|\mathfrak{Z}(U_t^+ j) - \mathfrak{Z}(j)\| \leq \|\mathfrak{Z}(U_t^+ j) - \mathfrak{Z}_n(U_t^+ j)\| + \|\mathfrak{Z}_n(U_t^+ j) - \mathfrak{Z}_n(j)\| + \|\mathfrak{Z}_n(j) - \mathfrak{Z}(j)\|$$

which may be made small for small t and large n .

Applying the foregoing results to the case of a Lie-group we may state

Proposition 3.3: Let U_t , $\mathbf{t} = (t_1, \dots, t_r)$, be a continuous, unitary representation of a r -dimensional Lie group G_r in \mathcal{H}_r , then $\pi(U_t)$ is a continuous, unitary representation of G_r in $\mathcal{L}^2(\mathcal{H}_r, \gamma)$.

To discuss the question if $\pi(U_t)$ is reducible we observe that $\pi(U)$ commutes with \mathbf{N} [cf. (2.16)] and thus leaves the eigen-spaces $\mathcal{L}_n^2(\mathcal{H}_r, \gamma)$ of \mathbf{N} invariant. Even the real and imaginary parts separately of the spaces \mathcal{L}_n^2 are stable against $\pi(U)$, since $\pi(U)$ maps a real-valued polynomial into one of the same kind.

If U is orthogonal

$$\mathbf{R} \pi(U) \mathfrak{Z}(j) = \int \mathfrak{Z}(U^+(j + i h)) \delta \gamma(h) = \langle 1, \mathfrak{Z}(U^+(j + i h))_h \rangle = \langle 1, \mathfrak{Z}(U^+ j + i h) \rangle_h = \pi(U) \mathbf{R} \mathfrak{Z}(j),$$

in accordance with Prop. 3.1. Hence,

$$\mathbf{R} \pi(U) \mathbf{R}^{-1} = \pi(U), \quad \text{for orthogonal } U. \quad (3.3)$$

Now $\pi(U)$ commutes with \mathbf{N} and \mathbf{R} and, therefore, with $\bar{\mathbf{N}} = \mathbf{R} \mathbf{N} \mathbf{R}^{-1}$, i. e.,

$$\pi(U) \bar{\mathbf{N}} = \bar{\mathbf{N}} \pi(U) \quad \text{for orthogonal } U. \quad (3.4)$$

Consequently, $\pi(U)$ leaves the real and imaginary parts of the eigenspaces $\bar{\mathcal{L}}_n^2(\mathcal{H}_r, \gamma)$ of $\bar{\mathbf{N}}$ also invariant. The representation $\pi(U_t)$ of the Lie group G_r is thus reducible in any way. When one is looking for the irreducible parts of $\pi(U_t)$, then one may start at once by reducing the restrictions of $\pi(U_t)$ to the mutually orthogonal subspaces $\bar{\mathcal{L}}_n^2(\mathcal{H}_r, \gamma)$.

Observing (2.18) one gets

$$\mathbf{V} \pi(U) \mathfrak{Z}(j) = e^{-(j, j)/2} \mathbf{D}_{\sqrt{2}} \mathfrak{Z}(U^+ j) = e^{-(U^+ j, U^+ j)/2} \mathfrak{Z}(U^+ \sqrt{2} j) = \pi(U) \mathbf{V} \mathfrak{Z}(j),$$

i. e.,

$$\mathbf{V} \pi(U) \mathbf{V}^{-1} = \pi(U) \quad \text{for orthogonal } U. \quad (3.5)$$

That means, that the transition to the space $\mathcal{L}^2(\mathcal{H}_r, \lambda)$ does not alter the structure of $\pi(U)$, U being orthogonal, and that $\mathcal{L}^2(\mathcal{H}_r, \lambda)$ decomposes into the direct sum of the invariant eigen-spaces of the operator $\bar{\mathbf{N}}_i = \mathbf{V} \bar{\mathbf{N}} \mathbf{V}^{-1}$.

In case that U is not an orthogonal transformation in \mathcal{H}_r — a case, which is also relevant for physical applications — things alter considerably. The relation $\pi(U^+) = \pi(U)^+$, which holds for orthogonal operators in virtue of Prop. 3.1 is no more valid in general, and the transformed operator-functions

$$\bar{\pi}(U) := \mathbf{R} \pi(U) \mathbf{R}^{-1} \quad (3.6)$$

and

$$\pi_i(U) := \mathbf{V} \bar{\pi}(U) \mathbf{V}^{-1} \quad (3.7)$$

do no longer coincide with $\pi(U)$. But $\bar{\pi}(U)$ is that group-representing operator-function, which is associated with the orthonormal Hermite polynomials and thus deserves some interest. It is, therefore, not at all astonishing that $\bar{\pi}(U)$ displays the nice *-homomorphism-property for all operators U . Rather surprising is, however, how involved the explicit expression is for that operator-function. And that, in spite of the very appealing structure of its infinitesimal generator.

In principle, $\bar{\pi}(U)$ is known if one has calculated its action on the basis-elements $\mathfrak{H}(k, j)$ for all $k \in K$. We have done this in the Appendix and received

$$\begin{aligned} \bar{\pi}(U) \mathfrak{H}(k, j) &= \sum_{\varrho_1=1}^{k_1} \cdots \sum_{\varrho_m=1}^{k_m} \left[(\sqrt{k_1!}/\varrho_1!) \|U e_1\|^{\varrho_1} H_{\varrho_1}((U e_1, j)/\|U e_1\|) \right. \\ &\quad \cdots \times (\sqrt{k_m!}/\varrho_m!) \|U e_m\|^{\varrho_m} H_{\varrho_m}((U e_m, j)/\|U e_m\|) \sum_{t \in T} ((-1)^{t_i} t_1^2!) \\ &\quad \times (U e_2, U e_1)^{t_i} ((-1)^{t_i} t_1^3!) (U e_3, U e_1)^{t_i} \cdots ((-1)^{t_{m-1}^m} t_{m-1}^m!) (U e_m, U e_{m-1})^{t_{m-1}^m} \left. \right] \end{aligned} \quad (3.8)$$

where m is chosen so large that $k_\mu = 0$, for $\mu > m$, and T denotes the set of $\frac{1}{2}m(m+1)$ -tuples of the shape $\mathbf{t} = (t_1^1, t_1^2, \dots, t_1^m, t_2^2, t_2^3, \dots, t_m^m)$ which satisfy the relations

$$k_1 - \varrho_1 = t_1^1, \quad k_2 - \varrho_2 = t_1^2 + t_2^2, \quad k_m - \varrho_m = t_1^m + t_2^m + \dots + t_m^m$$

and

$$t_1^1 - t_1^2 - \dots - t_1^m = 0, \quad t_2^2 - t_2^3 - \dots - t_2^m = 0, \quad t_m^m = 0.$$

If U is orthogonal ($U e_\mu, U e_\nu$) = 0 and $\|U e_\nu\| = 1$. Then in (3.8) only those terms are different from zero, for which $\mathbf{t} = 0$ and, consequently, $k_i = \varrho_i$, $i = 1, 2, \dots, m$, so that indeed $\bar{\pi}(U) = \pi(U)$. If $k_r = n \delta_{r1}$ (3.8) reduces to

$$\bar{\pi}(U) (1/\sqrt{n!}) H e_n((e_1, j)) = (\|U e_1\|^n / \sqrt{n!}) H e_n((U e_1, j) / \|U e_1\|). \quad (3.9)$$

In this special case we want to illustrate that $\bar{\pi}(U)$ is a *-homomorphism. For this purpose let us consider the scalar-product

$$\langle \bar{\pi}(U) H e_n((e_1, j)), H e_m((g, j)) \rangle \quad (i)$$

where $\|e_1\| = \|g\| = 1$. Only if $n = m$ and if g has a non-vanishing component g_1 along the $U e_1$ -axis (i) is unequal to zero. If g possesses also a component g_2 orthogonal to $U e_1$ we expand $H e_n((g_1, j) + (g_2, j))$ according to (A. 2) and observe that only the term with $H e_n((g_1, j))$ gives a non-vanishing contribution to (i). Thus, without restriction of generality, we may set $g = U e_1 / \|U e_1\|$. On the other hand we study the expression

$$\begin{aligned} \langle H e_n((e_1, j)), \bar{\pi}(U^+) H e_n((g, j)) \rangle &= \langle H e_n((e_1, j)), \|U^+ g\|^n H e_n((U^+ g, j) / \|U^+ g\|) \rangle \\ &= \lim_{m \rightarrow \infty} \int H e_n((e_1, j)) \|Q_m U^+ g\|^n H e_n((U^+ g, Q_m j) / \|Q_m U^+ g\|) \delta\gamma(j) \\ &= \lim_{m \rightarrow \infty} \int H e_n(x_1) (u_1^2 + \dots + u_m^2)^{n/2} H e_n((\sum_{r=1}^m u_r x_r) / (\sum_{r=1}^m u_r^2)^{1/2}) d\gamma_m, \end{aligned} \quad (ii)$$

where we have e_1 supplemented to an orthonormal basis $\{e_r\}_{r=1}^\infty$ of \mathcal{H}_r and have used the notations $Q_m = \sum_{\mu=1}^m e_\mu(e_\mu, \cdot)$, $(e_r, j) = x_r$, and $(U^+ g, e_r) = u_r$. Applying once again (A. 2) (ii) takes the simple form

$$\begin{aligned} (ii) &= \lim_{m \rightarrow \infty} \int H e_n(x_1) u_1^n H e_n(x_1) d\gamma_m \\ &= \int \|U e_1\|^n H e_n((U e_1, j) / \|U e_1\|) H e_n((g, j)) \delta\gamma(j) = (i) \end{aligned}$$

since $u_1 = \|U e_1\|$, and one may interpret the bounded variable x_1 equally well as (g, j) . Thus we have shown that

$$\bar{\pi}(U^+) = \bar{\pi}(U)^+ \quad (3.10)$$

in that peculiar example. That (3.10) is true in general is a consequence of a Proposition of Part II.

As for $\bar{\pi}_\lambda(U)$, it is easy to show that (3.8) remains valid if one replaces $\bar{\pi}$ by $\bar{\pi}_\lambda$, $\mathfrak{S}(k, j)$ by $\mathfrak{S}_\lambda(k, j)$, and $H e_n((e_r, j))$ by

$$\mathfrak{S}_n((e_r, j)) := \exp[-(j, j)/2] \mathbf{D}_{\sqrt{2}} H e_n((e_r, j)).$$

The relation

$$\bar{\pi}_\lambda(U^+) = \bar{\pi}_\lambda(U)^+ \quad \text{follows from (3.10) and unitarity of } \mathbf{V}. \quad (3.11)$$

Let us now — as in Proposition 3.2 — consider a weakly continuous operator-function U_t defined on the dense domain $\mathcal{D} \subset \mathcal{H}_r$. Then $\bar{\pi}(U_t)$ is strongly continuous on the dense domain $\mathcal{P}_\mathcal{D}$.

For, take a $\mathfrak{L}(j) \in \mathcal{P}_\mathcal{D}$, then

$$[\bar{\pi}(U_t) \mathfrak{L}](h) = [\mathbf{R} \pi(U_t) \mathbf{R}^{-1} \mathbf{R} \mathfrak{L}](h) = \langle 1, \mathfrak{S}(U_t^+ h + i U_t^+ j) \rangle_j, [\mathfrak{S}(j) \in \mathfrak{P}_\mathcal{D}].$$

But the operator-function $W_t^+ j = U_t^+ h + i U_t^+ j$ is also weakly continuous and according to Prop. 3.2 $\mathfrak{S}(U_t^+ h + i U_t^+ j)$ tends strongly to $\mathfrak{S}(h + i j)$, for $t \rightarrow 0$.

By the continuity of the scalar product we, therefore, obtain

$$\lim_{t \rightarrow 0} \bar{\pi}(U_t) \mathfrak{L}(h) = \lim_{t \rightarrow 0} \langle 1, \mathfrak{S}(U_t^+ h + i U_t^+ j) \rangle_j = \langle 1, \mathfrak{S}(h + i j) \rangle_j = \mathbf{R} \mathfrak{S}(h) = \mathfrak{L}(h).$$

Since \mathbf{V} is bounded $\bar{\pi}_\lambda(U_t)$ is continuous, too.

Let us summarize these results into

Proposition 3.4: The operator-functions $\bar{\pi}(U)$ resp. $\bar{\pi}_\lambda(U)$ are strongly continuous *-homomorphisms with respect to operator-multiplication and Hermitian conjugation.

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Appendix

We here present the addition-theorem for Hermitian polynomials, referred to several times in the text and slightly more generalized than in ¹³. Starting from

$$\exp(zx - z^2/2) = \sum_{n=0}^{\infty} H e_n(x) z^n/n! \quad (\text{A. 1})$$

we obtain

$$H e_n(a_1 x_1 + \dots + a_m x_m / (a_1^2 + \dots + a_m^2)^{1/2}) = \frac{d^n}{dz^n} \left[\exp \left[z \left(\sum_{\mu=1}^m a_\mu x_\mu / w \right) - z^2/2 \right] \right]_{z=0},$$

where $w = (a_1^2 + \dots + a_m^2)^{1/2}$,

$$\begin{aligned} \dots &= \frac{d^n}{dz^n} \left[\prod_{i=1}^m \exp \left[(z a_i x_i / w) - (a_i^2 z^2 / w^2 2) \right] \right]_{z=0} \\ &= \sum_{\substack{m \\ \sum_{i=1}^m \mu_i = n}} \frac{n!}{\mu_1! \dots \mu_m!} \prod_{i=1}^m \frac{d^{\mu_i}}{dz^{\mu_i}} \left[\exp \left[(z a_i x_i / w) - (a_i^2 z^2 / w^2 2) \right] \right]_{z=0}. \end{aligned}$$

Thus,

$$H e_n \left(\frac{a_1 x_1 + \dots + a_m x_m}{\sqrt{a_1^2 + \dots + a_m^2}} \right) = (a_1^2 + \dots + a_m^2)^{-n/2} \sum_{\substack{m \\ \sum_{i=1}^m \mu_i = n}} \frac{n!}{\mu_1! \dots \mu_m!} \prod_{i=1}^m a_i^{\mu_i} H e_{\mu_i}(x_i). \quad (\text{A. 2})$$

We now supplement the calculation leading to (3.8). We employ the notation $\partial_\mu^{k_\mu} := \partial^{k_\mu} / \partial z_\mu^{k_\mu}$.

$$\bar{\pi}(U) \mathfrak{H}(k, j) = \mathbf{R} \pi(U) \mathfrak{P}(k, j) = \mathbf{R} \pi(U) \prod_{\mu=1}^m (k_\mu!)^{-1/2} \partial_\mu^{k_\mu} [\exp(\sum_{\mu=1}^m z_\mu (e_\mu, j))]_{z_\mu=0},$$

where m is chosen so large that $k_\mu = 0$ for $\mu > m$.

Now $\mathbf{R}(g, j)^n = \|g\|^n \mathbf{R}(\hat{g}, j)^n = \|g\|^n H e_n((\hat{g}, j))$,

where $\hat{g} = g/\|g\|$, cf. also (2.13). Thus,

$$\mathbf{R} \exp[(g, j)] = \mathbf{R} \sum_{n=0}^{\infty} (1/n!) (g, j)^n = \sum_{n=0}^{\infty} (\|g\|^n/n!) H e_n((\hat{g}, j)) = \exp[(g, j) - \|g\|^2/2]$$

according to (A. 1).

Hence,

$$\begin{aligned} \bar{\pi}(U) \mathfrak{H}(k, j) &= \prod_{\mu=1}^m (k_\mu!)^{-1/2} \partial_\mu^{k_\mu} [\exp(\sum_{\mu=1}^m z_\mu (U e_\mu, j) - \sum_{\mu, r=1}^m z_\mu z_r (U e_\mu, U e_r)/2)]_{z_\mu=0} \\ &=: \prod_{\mu=1}^m (k_\mu!)^{-1/2} \partial_\mu^{k_\mu} [E(\mathbf{z}) \exp\{q(\mathbf{z})\}]_{\mathbf{z}=0}, \end{aligned} \quad (\text{i})$$

where

$$\mathbf{z} = (z_1, \dots, z_m), E(\mathbf{z}) = \exp\left\{\sum_{\mu=1}^m [z_\mu (U e_\mu, j) - z_\mu^2 (U e_\mu, U e_\mu)/2]\right\},$$

and

$$-q(\mathbf{z}) = \sum_{\mu \neq \nu} z_\mu z_\nu (U e_\mu, U e_\nu) / 2.$$

We have

$$\prod_{\mu=1}^m \partial_{\mu}^{k_\mu} (E(\mathbf{z}) \exp\{q(\mathbf{z})\}) = \sum_{\varrho_1=1}^{k_1} \cdots \sum_{\varrho_m=1}^{k_m} [\binom{k_1}{\varrho_1} \cdots \binom{k_m}{\varrho_m} E^{(\varrho_1, \dots, \varrho_m)}(\mathbf{z}) \partial_1^{\sigma_1} \cdots \partial_m^{\sigma_m} \exp\{q(\mathbf{z})\}], \quad (\text{ii})$$

where $\sigma_i = k_i - \varrho_i$. By complete induction one gets

$$\begin{aligned} \partial_m^{\sigma_m} \cdots \partial_1^{\sigma_1} e^q &= \sum_{\substack{m \\ \sum_{i=1}^m t_i^m = \sigma_m}} \cdots \sum_{\substack{2 \\ \sum_{i=1}^2 t_i^2 = \sigma_2}} \sum_{t_1^1 = \sigma_1} \frac{\sigma_m!}{t_1^m! \cdots t_m^m!} \frac{\sigma_2!}{t_1^2! t_2^2!} \frac{\sigma_1!}{t_1^1!} \frac{t_1^1!}{(t_1^1 - t_1^2 - \cdots - t_1^m)!} \cdots \\ &\quad \times \frac{t_{m-1}^{m-1}}{(t_{m-1}^{m-1} - t_{m-1}^m)!} (\partial_1 q)^{t_1^1 - t_1^2 - \cdots - t_1^m} (\partial_m \partial_1 q)^{t_1^m} \\ &\quad \cdots (\partial_2 \partial_1 q)^{t_1^m} (\partial_2 q)^{t_2^2 - t_2^3 - \cdots - t_2^m} (\partial_m \partial_2 q)^{t_2^m} \cdots (\partial_3 \partial_2 q)^{t_2^{3m}} \cdots (\partial_m q)^{t_m^m} e^q. \end{aligned} \quad (\text{iii})$$

At the point $\mathbf{z}=0$, where $q=0$, survive in (iii) only those terms for which vanish all exponents t of the powers $(\partial_i q)^t$, since the $\partial_i q$ are equal to zero, too. That gives the conditions

$$t_1^1 - t_1^2 - t_1^3 - \cdots - t_1^m = 0, \quad t_2^2 - t_2^3 - \cdots - t_2^m = 0, \quad \dots, \quad t_m^m = 0, \quad (\text{iv})$$

which are to hold simultaneously with

$$\sigma_1 = t_1^1, \quad \sigma_2 = t_1^2 + t_2^2, \quad \sigma_m = t_1^m + t_2^m + \cdots + t_m^m \quad (\text{v})$$

originating from the summation in (iii). By (iv) and (v) a set T of $m(m+1)/2$ -tuples $\mathbf{t} = (t_1^1, t_1^2, \dots, t_m^m)$ is defined. At $\mathbf{z}=0$ follows from (iii)

$$\begin{aligned} \partial_m^{\sigma_m} \cdots \partial_1^{\sigma_1} e^q|_{\mathbf{z}=0} &= \sum_{\mathbf{t} \in T} \frac{\sigma_1! \cdots \sigma_m!}{t_1^2! t_1^3! t_2^3! \cdots t_{m-1}^m!} (-1)^{t_1^m} (U e_m, U e_1)^{t_1^m} \cdots \\ &\quad \times (-1)^{t_1^2} (U e_2, U e_1)^{t_1^2} (-1)^{t_2^m} (U e_m, U e_2)^{t_2^m} \cdots (-1)^{t_2^3} (U e_3, U e_2)^{t_2^3} \cdots (-1)^{t_{m-1}^m} (U e_m, U e_{m-1})^{t_{m-1}^m}. \end{aligned}$$

On the other hand at $\mathbf{z}=0$ we obtain

$$E^{(\varrho_1, \dots, \varrho_m)}(0) = \prod_{\mu=1}^m \partial_{\mu}^{\varrho_{\mu}} \{ \exp[z_{\mu} (U e_{\mu}, j) - z_{\mu}^2 \| U e_{\mu} \|^2 / 2] \} |_{\mathbf{z}=0} = \prod_{\mu=1}^m H e_{\varrho_{\mu}} ((U e_{\mu}, j) / \| U e_{\mu} \|) \| U e_{\mu} \|^{2\varrho_{\mu}}.$$

Altogether we arrive at

$$\begin{aligned} \bar{\pi}(U) \mathfrak{H}(k, j) &= \sum_{\varrho_1=1}^{k_1} \cdots \sum_{\varrho_m=1}^{k_m} (V \overline{k_1!} / \varrho_1!) \| U e_1 \|^{2\varrho_1} H e_{\varrho_1} \left(\frac{(U e_1, j)}{\| U e_1 \|} \right) \cdots \\ &\quad \times (V \overline{k_m!} / \varrho_m!) \| U e_m \|^{2\varrho_m} H e_{\varrho_m} \left(\frac{(U e_m, j)}{\| U e_m \|} \right) \sum_{\mathbf{t} \in T} ((-1)^{t_1^2} / t_1^2!) (U e_2, U e_1)^{t_1^2} \quad (\text{A. 3}) \\ &\quad \times ((-1)^{t_1^3} / t_1^3!) (U e_3, U e_1)^{t_1^3} \cdots ((-1)^{t_{m-1}^m} / t_{m-1}^m!) (U e_m, U e_{m-1})^{t_{m-1}^m}. \end{aligned}$$